To Major. G. F. Jenks Ordnance
Corps U.S. Army, with
the Compliments of the
Author

"MATHEMATICAL MICROBES."

A PAPER

READ FOR

"THE CLUB" (LITERARY)

— OF —

SPRINGFIELD, MASS.

FEBRUARY 11th, 1898

— BY —

A. R. BUFFINGTON,

AN EX-MEMBER

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1898
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MATHEMATICAL MICROBES.

I call the subject of this paper "Mathematical Microbes," as the quantities involved are microscopical, but essential to exactness—as essential as the discovery of microbes to correct diagnosis of disease, assuming that every disease has its distinctive microbe. You are to judge whether it is a diseased mathematics to which I call your attention to-night.

In the field of criticism, it seems to me "common sense" judgment is either altogether excluded or ungraciously tolerated. By "common sense," let us understand that intuitive knowledge by which men generally pronounce judgment on subjects and things in the knowledge of which they have not been specifically educated. A judgment based on common sense—on "The eternal fitness of things"—does not frequently accord with the technique of a cult or a science. To be specially educated, let us say, to skilfully handle the tools of those who have gone before in any calling, ought not to darken the discernment of any truth pertaining to it. The wider and greater the knowledge the more tolerant of special ignorance it should be, for the wisest is but a child, and his wisdom often more a matter of chance than ability. Chance has played a greater role in the achievements of men than is commonly understood. To see nothing that does not fit into the moulds that shaped their attainments in any branch of knowledge may not apply to the pioneers—the originators, the high priests, whether by sheer ability or by accident—of knowledge, but it seems to stamp the merely learned in what other men have accomplished. It is these who shape the opinions of less learned men and inoculate them with intolerance.

Now, common sense may be truer even than mathematics, and when combined with the genius *chance*, may accomplish what mathematics cannot in its own proper field; that is, eradicate error and establish truth. Let it be understood that you owe this paper to chance, and that the preceding *wise* remarks are merely to pave the way for a statement of how it happened, and what followed. As a student I accepted my mathematics as I accepted my religious faith—both, as presented to me by books and teachers, accorded with my discernment of truth. I no more thought of questioning one than the other and in this desirable frame of mind, so far as mathematics is concerned, I continued until quite recently.

Two or three years ago I had, in some of my mechanical work, evolved a peculiar triangle whose solution I deemed it desirable to have, and I set about it in the ordinary way, but this way proved unsatisfactory, as it involved the extraction of the square root of imperfect squares. I wanted perfect values, not approximations, but to get them I needed relations, or equations, not based on the Pythagorean law that the square of the hypothenuse of a right angle triangle equals the sum of the squares of the other two sides. To avoid the square root, I required an equation involving but the first power of the variable, or unknown quantity. Such an one would, of course, give values freed from the inaccuracies of the square root of imperfect squares. In my efforts to obtain this I discovered that in subtracting an algebraic expression from one member of an equation I had evolved, there was no difference in result; that is, whether it were subtracted or not the resulting values were not affected. Now, there could be but one meaning for such a result, and that was, that this algebraic expression must be nothing, must be zero, for only zero subtracted from a thing could give a difference equal to the thing itself. It, the said expression, contained but the first power of the unknown quantity, and, as explained, being zero, if equated with zero,—it's equal,—would furnish the equa-

tion I was seeking. Accident had, apparently, given me what I had failed to work out and I thus found the values sought without extracting the square root of imperfect squares. But I was confronted with something that shook my mathematical faith to the foundation, viz: if these values were true the Pythagorean law was not for the particular triangle involved. How could this be so, for the proof—geometrical proof—of that law is *general*, not particular? It applies to *any* right angle triangle, how then can there be an exception? In the face of so perfect a proof who would dare proclaim an exception? The old apothegm that "The exception proves the rule" would not apply in this case, and I should have to establish it—the exception—by well known and accepted mathematical methods before it would be received by any one. This I found to be no easy task, but I labored at it faithfully and have now arrived at a demonstration satisfactory, at least, to myself.

In the *Atlantic Monthly* for August, 1897, is a paper entitled, "The pause in criticism—and after," in which occurs this statement:

"Every so-called law was originally only the opin-"ion of one man."

Again. "When knowledge has reached the stage "where it can be packed into formulas one of two "things happens: either the formulas are easily learned "and repeated mechanically, which leads to petrifac-"tion, or they serve as new points of departure from "which the untrammelled" mind "sets out on a higher "quest."

As an illustration of the former case rhetoric is used, and then the paper adds: "How different is the "aspect of those sciences and arts in which classifica-"tion neither implies arrested development nor marks "the limit beyond which progress cannot be made! "We need cite as an illustration only the mathematics, "one of the branches of knowledge in which fixed laws "were earliest formulated, and the science above all "others in which absolute accuracy can be attained at

"every step; age, for it, does not mean senility; rules "are not shackles." To this the majority will agree, not only as to the present, but the past aspect of mathematics. Yet even this may fall within the domain of the criticism; that is, it may not be exempt from classification with the inexact sciences packed into formulas, easily learned and repeated mechanically, not only leading to petrifaction, but constituting completed forms of petrifaction.

To a more or less limited extent the old aphorism that "Truth is many sided" may apply even to mathematics. Among its formulas there may be some that comprehend but one face of truth, rejecting all others.

In these latter days of a marvelous century there are evidences, not hard to find, that man has, to a large extent, freed himself from the shackles of environment, education and the consequent prejudices, sufficiently even to listen with some degree of patience to questionings of mathematical truth. Knowledge grows by ever widening circles. Time has long since passed, when a scientist, so called, was "a jack of all trades" and perfect in all; when a college professor occupied several chairs and felt competent to instruct in any branch of knowledge.

So vast has become the field in which the searchers for truth are working that it may be said, without exaggeration, that man's short life may be more than filled by the study of a single drop of water. Not only new fields are entered but old ones are gleaned over for grains of truth that may have escaped the first reapers.

It is into one corner of one of these old fields I wish to take you for a search after, perchance, a lost grain, overlooked—hidden—in the long ages since Pythagoras announced the law I have stated,—which is the very corner stone of mathematics. This law, ever since enunciated, has been accepted by "all sorts and conditions of men" as an unchanging and unchangeable truth, absolutely without an exception. To question its universality—to say that it has exceptions—would

appear to be, prima facie, evidence of an unbalanced mind, or, of a temerity bordering on the marvelous.

While accepting the positive value of the proof—in fact, presenting as I shall myself, a proof more positive and convincing if possible than the proofs found in the books—I have mathematical ground for questioning the universality of the law, and in placing it before you I beg to say that it involves no abstruse paths hard to follow. There is nothing in it beyond the four rules of arithmetic and the extraction of the square root—nothing but what any school boy can follow and what he may read as he runs. Yet, the hardest to prove is the simplest truth! One truth ends where another, which it *cannot* include, begins. This may be the basis of the saying that "the exception proves the rule." A law or rule to be universal can have no exception and the law of Pythagoras, as proved, admits of no exception, it is therefore universal if the proof be admitted. *That* the square of the hypothenuse of a right triangle equals the sum of the squares of the other two sides is a law, and one truth, does not annul other truths. The mere statement of it involves another law, and states another law which it cannot make void should it in any case conflict with it. Both laws must stand inviolate with reference to each other. To say that one thing equals another, which the Pythagorean law does, means an equation—states an equation—the members of which are interchangeable and absolutely equal. Whatever one is, the other either must be, or readily, from the existing equality of values, be convertible into it. $a=b$ is an equation and when the values of the letters be substituted for them, one member becomes but a repetition of the other. If "a" expresses in algebraic terms a geometrical figure, "b" must be the same, or the equal in area of this figure, and when this area is expressed in numbers of *some* unit or subdivisions of it, the area of "a," whatever be the figure represented, must be the same. If "a" represent a perfect square, geometrically and algebraically, "b," however expressed, must also be the equal in area

of this perfect square and geometrically convertible into it. In short, one member of an equation cannot be even infinitesimally less or greater than the other ; also if one be a positive, determinable figure or quantity the other must be the same—one cannot be a transcendental, incommensurable or indefinite quantity and the other a positive, definite and commensurable one without making void the truth the equation stands for, or expresses. This is what appears to take place when the Pythagorean law is made universal ; one face of truth is recognized to the prejudice of another equally as important and equally as powerful for the establishment of correct knowledge—for the maintenance of that exactness without which mathematics becomes unreliable. This, I take it, is common sense—is a common sense judgment. That *any* two perfect squares added together will produce a perfect square, in the nature of things strikes one as a statement that must have a proof devoid of any weak point. Now, when we leave the geometrical proof—which, so far as I know or any other person known to me knows, has no weak point—and (substitute for things themselves ~~for~~ their respective symbols) we must be satisfied—except in all cases involving what have been rightly called "right angle triangle numbers"—with equations one member of each of which is covered by the "radical sign," meaning the extraction of the square root. The square roots of the first members of these equations have been extracted : they were perfect squares, but their conceded equals, now under the radical sign, are not perfect squares, tho' members of equations, proclaiming perfect, not approximate, equality with perfect squares. It strikes me there must be something wrong with this state of things. Symbols, to be symbols at all, must equal the things represented or they are false. To say let x, or y, or z, represent this line or that line, or this square or rectangle, or that square or rectangle, means that the person who says so stipulates that in all he proposes to do with x, y, or z; x, y, or z shall, not may, truthfully and unequivocally represent the said line,

And substitute for things
themselves their respective
symbols

square, or rectangle. It is convenient, simplifies, and saves labor for him or for those he wishes to inform, or argue with, to use symbols for the things themselves, but if he be held to a strict account in the use of them, he cannot produce results that would not have been arrived at had the things themselves been used. If he does, then we have a right to question the truth of the results—he has not kept faith with his own agreement. Ah, but mathematicians will say, the things themselves are not commensurable—there is no common unit, hence the approximation of the square root. Numbers are not flexible, not divisible, and cannot be used where their symbols can be; cannot be made to follow their symbols thro' the labyrinth of mathematical processes—algebra is not arithmetic. Arithmetic is well enough, but it is limited; when we go into abstruse, complicated mathematical processes numbers cannot be used, we must have some other tools. These enable us to make algebraic squares of the first members of the equations you are criticising. In the untrammelled region of mind—the boundless field wherein man's mind may work in abstruse speculation or mathematical work—realities, numbers, have no place. These would clog the work; no useful results could be produced. This sounds like good logic; it satisfies the mathematician and he makes it satisfy other men. The syllogisms are all right if the premises be granted, but the premises I challenge. The things themselves are not commensurable, it is said, if approximations be produced by combining them. There never was, it seems to me, either actually visible, tangible and existing, or conceived of in the "boundless field wherein man's mind may work," a line, surface or solid, that is not commensurable with any other line, surface, or solid so existing or conceived of, because they are all infinitely divisible. There is no limit to the division of the things compared and it can be carried on until a common unit can be found; when that is reached the things are commensurable. This statement, or premise, therefore, amounts to nothing more than this, " we get

these incomplete squares for the second members of the equations referred to because we have not a common unit for the lines, &c., dealt with." Thus the mathematician has not been able to find a common unit and has been obliged to make one for himself—the decimal—with which he makes the world to square. When he finds two things incommensurable—that is, not commensurable within the method used for extracting the square root, he applies to the problem the decimal division of the unit. So poor is he that he has only one out of the infinite number into which the unit may be divided. This brings me to the second statement or premise, viz.: that numbers are not flexible and cannot be made to follow their symbols in labyrinthian mathematical processes. This statement has also been accepted—has been granted to the mathematician. Numbers and the units of numbers are as flexible as their symbols can be. They can be carried thro' as many and complicated mathematical gymnastics as their symbols or representatives; no more, no less. They are infinitely divisible and their units are infinitely divisible; so much and no more can be said of their symbols. Besides symbols—representatives—if they are capable of but one mathematical summersault beyond what the things represented are capable of, they cease to be symbols or representatives of these things.

With so-called square roots, and the method of obtaining them I have no quarrel. The square root, if imperfect, approximates to something that does not exist. It's the nearest approach that man has yet made to a creation. He produces by algebraic process something, or thinks he does, he calls a square, which is not a square, because there is nothing in nature that will represent or express a side of it, and then gravely proceeds to extract its square root, which on inspection is found as mythical as his square. He cannot construct his square and should he live forever he could not reach in his process—tho' it be scientific—a completed line to represent a side of it. Both of them—square and line

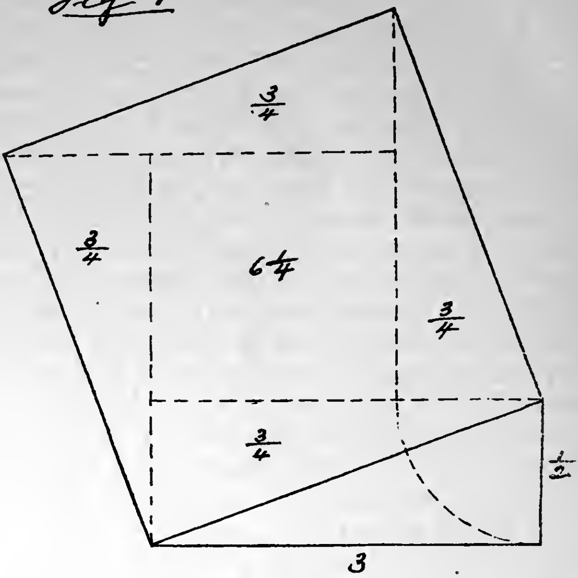
—are the purest fiction and this is called mathematics! That a given area—note that I say *given*, that is, actually existing—can be represented approximately by one of the infinite number of squares that can be made from the infinite number of sides that can be produced by extracting the square root of the number expressing the area, is a truth so far as it goes, but we know what it is when we get it and we know how we got it. There is no questionable point in the process. It is an approximation; it's what we started out to get, and it is limited only by the number of decimals at which we stop in the process of extracting the square root. The process itself is not only scientific but it is a beautiful one—one that enables us readily to get the square root of perfect squares, as well as approximations for unperfect squares.

The existing algebra, trigonometry, and even the differential calculus are permeated with the fictions, the method of the production of which I have pointed out. It is true these fictions are approximations and differ from the true values by microscopical quantities, and for all ordinary purposes satisfy requirements. For astronomical calculation involving inconceivable distances they might make appreciable differences.

In a paper of this kind I can but give a mere outline of the detailed proof of the existence of the small quantitives that have been neglected. Before doing so I will give a proof of the Pythagorean law so concise and convincing that there would appear to be no escape from the conclusion.

Observe this triangle—right angle triangle:

Fig 1



It is the geometrical embodiment of the algebraic statement that the sum of the squares of two quantities equals the sum of the square of their difference and twice their rectangle. Thus $a^2+b^2=(a-b)^2+2ab$. This simple algebraic law is only another statement of the Pythagorian, and is its perfect proof. You see in the figure the square of the difference of the base and perpendicular surrounded—symmetrically and completely filling the remaining area of the square on the hypothenuse—by four right angle triangles exactly equal to the given one and the sum of them equal to twice the rectangle of the base and perpendicular of it—the whole together being the square on the hypothenuse, which is therefore equal to the sum of the squares of the other two sides of the given triangle. Now if you give these sides values you can place the values of the parts, into which the square on the hypothenuse is divided, on them and the sum proclaims at once that this square equals the sum of the squares of the other two sides. Call the

base 3 and the perpendicular $\frac{1}{2}$, the difference is $2\frac{1}{2}$, the square of which is $6\frac{1}{4}$. The area of the given triangle is $3\times\frac{1}{2}\div 2=\frac{3}{4}$. Four times $\frac{3}{4}$ is 3, which equals twice the rectangle of the two sides. Thus the eye as well as the understanding takes in the truth of the Pythagorean law.

The particular triangle I have mentioned as resulting from some work in which I was engaged, was peculiar in that but one side—the base—and part of the hypothenuse were known, with which to determine the other parts.

Observe this figure (Fig. 2) which presents this triangle:

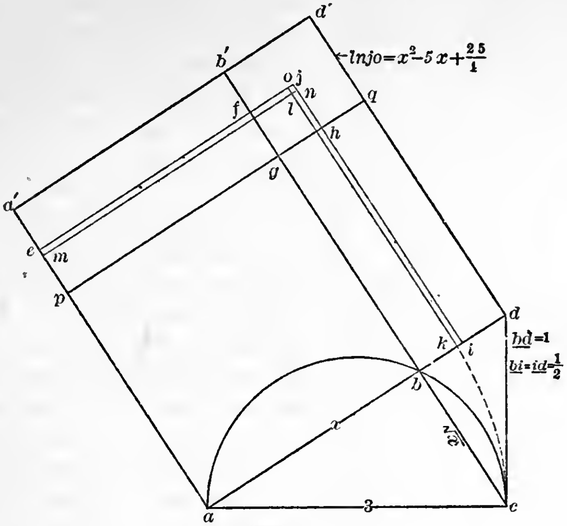


FIG. 2.

The figure shows the square erected on the hypothenuse of the triangle. The base *ac* is 3 and that part of the hypothenuse, *bd*, outside of the circle described on the base, is 1. The line, *cb*, we know, from one of the properties of the circle, is perpendicular to the hypothenuse at the point *b*, and extended up thro' the square on the hypothenuse, it divides it into two rectangles, the larger one, by the Pythagorean law, being equal to the

square on the base; the smaller, equal to the square on the perpendicular. The square erected on $\overline{ab}$, the unknown part of the hypothenuse, divides the larger rectangle into two parts—a perfect square and a rectangle, the side $\overline{pa'}$ of the latter being equal to $\overline{bd}$, the known part of the hypothenuse, and if the upper half of it, *a'efb'*, be removed and placed in the position, *bihg*, it will fill half the space *gbdq*. Thus the whole of the larger rectangle will become the irregular figure *efghiba*—that is, a square, lacking but a corner, which, if filled in, will make it perfect, and we see that this corner is the square of $\overline{bi}$—half of $\overline{bd}$. We have thus completed, geometrically, the square of the larger rectangle and if it equal the square on the base, the same quantity added to the square on the base will make the sum equal to the completed square of the larger rectangle. Now, if the base, $\overline{ac}$, with *a* as a pivot, be swung up into the hypothenuse, it will take the position $\overline{ak}$ and on it we will erect the square *makl*—in other words, the square on the base $\overline{ac}$ has been transferred to a new position on the hypothenuse, so that we have it inside of the completed square of the larger rectangle. The space—area—between the two, outside of one and inside of the other, must equal the addition *fghj*—the square of $\overline{bi}$—which we added to the larger rectangle to complete its square, *if* the square on the base be equal to the larger rectangle. If this space between the squares, which we see is composed of a minute square and two small rectangles, be not equal to the square of $\overline{bi}$, the Pythagorean law for this particular triangle cannot be true.

Having given the geometry of the matter we will now go into the algebra of it and between the former and latter, you will observe, there is perfect accordance, already shown in the proof I have given of the Pythagorean law, according to which $(\overline{ab})^2+(\overline{bc})^2=(\overline{ac})^2=9$, as $\overline{ab}$ and $\overline{bc}$ are respectively the base and perpendicular of the right angle triangle *abc*. Let us call $\overline{ab}$, x, $\overline{bc}$ will, under the law, be a mean proportional between $\overline{ab}$ and $\overline{bd}$; that is, its square will equal $\overline{ab}\times\overline{bd}$, and $\overline{bc}$ itself equal $\sqrt{ab\times bd} = \sqrt{x\times 1} = \sqrt{x}$. Substituting for $(\overline{ab})^2$

and $\underline{(bc)}^2$ their symbals, thus assumed and determined, under the law, we have the equation

$$x^2+x=9\ldots.(1)$$

apparently at first the only equation that could be constructed for finding the values of the unknown parts of the given triangle. To do this from this equation, the first member x^2+x, altho' the equal—so made—of a perfect square (9), is not one algebraically and must be made one. This is done by taking half of the coefficient of the first power of x, squaring it and adding it. This coefficient is 1 (which we see is the value of $\underline{bd}$, with which we multiplied $\underline{ab}$ to get an expression for $\underline{bc}$), half of it, is ½ and this squared is ¼, which added to the first member of equation (1) makes it $x^2+x+\frac{1}{4}$. This addition is inadmissable unless the same be made to the second member, altho' it's a perfect square already and by the law of the equation should equal an area in the first member readily convertible into a perfect square.

The equation thus becomes:

$$x^2+x+\tfrac{1}{4}=9\tfrac{1}{4}.\ \ .\ \ .\ (2)$$

and we are confronted with the same state of things that existed in the equation from which we started. That is, we have one member a perfect square whereas the other is not, but in the last case the responsibility is shifted from algebra to arithmetic. In the first case arithmetic was equal to the extraction of the square root, but algebra couldn't do its part; now algebra can, but arithmetic cannot. However, we can do now, what we could not do before, we can *force* a solution. What we could not do with 9's equal, that is, its symbol or representative, we can make *9* itself—or rather *9* now loaded down by a $\frac{1}{4}$—do. So much for the flexibility of symbols. We see, however, in the operation the perfect harmony that exists between algebra and geometry and we are not left only able to follow the gymnastics of numbers, or things, in mathematical work; we can at least, thro' the eyes of geometry, have some idea of what algebra is doing and possibly in that way, keep it out of mischief. Solving the last equation by extracting the square root of both members we arrive

Symbols or Representations

at $x+\frac{1}{2}=\frac{\sqrt{37}}{2}$ or $x=\frac{\sqrt{37}-1}{2}=2.54138+ + +$ to infinity—a value very much resembling a tapeworm, the only essential part about it for its existence being its head. The tail you can chop off at any place suiting your convenience at the time, but the head remains and will live even if deprived of its last decimal. We have found something for the second member of a new equation which we call the value of x; that is of the side $\underline{ab}$ and to get $\underline{bc}$ we must put what is left of this poor value thro' the cruel process of going again thro' the decimal shambles to come out an unrecognizable abstraction representing nothing in this world or the next; the x, however, and the $\sqrt{x}$, you observe, remain unmutilated—only the despised arithmetic suffers; the things, not the symbols, suffer. Referring again to the figure 2 we are concerned with the two small rectangles and the minute square, to which attention has been called. In whatever way it is attempted to make, or find, relations between these, themselves, or between them and the other parts, there will result either the expression $x^2+x-\frac{35}{4}$ or $\frac{1}{4}$ as the value of their sum; showing that the value of the area between the square 9 and the completed square of the larger rectangle is either $x^2+x-\frac{35}{4}$ or $\frac{1}{4}$ depending upon how it is done. If $x^2+x-\frac{35}{4}$ be equated with $\frac{1}{4}$, to which the law *makes* it equal, it results in returning to the original equation $x^2+x=9$ which means nothing more than that we have forced that relation—it does not mean that it is true, if the law be questioned. An examination of the method by which the $\frac{1}{4}$ is determined as the sum of the parts of the area between the squares will fix the parts of which this $\frac{1}{4}$ is the sum. The small line $\underline{ki}$ is the side of the minute square and is equal to $\underline{ab}+\underline{bi}-\underline{ak}$; that is, $x+\frac{1}{2}-3=x-\frac{5}{2}$. Squaring this, we have the little square, as algebraically expressed, $x^2-5x+\frac{25}{4}$. Multiplying $\underline{ki}$ by $\underline{kl}$, that is, $x-\frac{5}{2}$ by 3, and then again by 2 we get $6x-15$ for the algebraic value of the two small rectangles. Now if in the expression for the little square, the value of x^2, obtained from the equation $x^2+x=9$, that is, $9-x$, be substituted for x^2, the expression becomes $\frac{61}{4}-6x$.

Thus we obtain the algebraic expressions for the sum of the small rectangles and the little square, containing but the first power of the variable x. Adding the two together, $\frac{61}{4}-6x+6x-15$,the sum is $\frac{1}{4}$, but examining the two expressions we see that this would be the sum no matter what value be given to x; that this sum is entirely independent of the value of the variable. Therefore an infinite number of values can be placed in these expressions for x and the sum of them will remain the same, and if there be anything wrong with them the trouble must enter into the value $(\frac{61}{4}-6x)$, for the little square when the substitution of $9-x$ is made for x^2.

Manifestly if $\frac{61}{4}-6x=x^2-5x+\frac{25}{4}$, the $\sqrt{\frac{61}{4}-6x}$ must equal $x-\frac{5}{2}$. We have an equation then, $\sqrt{\frac{61}{4}-6x}=x-\frac{5}{2}$. Now both members of an equation can be multiplied by the same quantity without affecting their equality. Multiply both members of this equation by the first, it becomes $\frac{61}{4}-6x=(x-\frac{5}{2})\sqrt{\frac{61}{4}-6x}$. Square both members to get rid of the radical and we finally have $(\frac{61}{4}-6x)^2=(x-\frac{5}{2})^2\times(\frac{61}{4}-6x)$—a perfectly regular and legitimate equation. Squaring the parts, as indicated, and reducing, result in an equation of the third degree, viz;

$$24x^3=37x^2+277x-549 \quad . \quad . \quad . \quad (3)$$

having a remarkable property. Two values of the variable will satisfy it, viz; $x=\frac{\sqrt{37}-1}{2}$ and $x=\frac{61}{24}$, the former being the value demanded by the equation, $x^2+x=9$, the latter a value that reduces the expression for the little square (after substituting in it $9-x$ for x^2) to zero; that is, $\frac{61}{4}-6x=0$. This fact requires the other part entering into the sum $\frac{1}{4}$—the part expressing the sum of the two little rectangles—to equal $\frac{1}{4}$; that is, $6x-15=\frac{1}{4}$. Hence the sum $6x-15+\frac{61}{4}-6x=\frac{1}{4}$, as demanded by the condition that $x^2+x=9$, and so the space between the square 9 and the completed square of the larger rectangle ~~on~~ of the square on the hypothenuse must equal $\frac{1}{4}$, as demanded by the Pythagorean law! Again taking for the expression $(\frac{61}{4}-6x)$ for the little square, its equal in two rectangles on two sides of the

+ of the squares &c.

square 9, or in one twice as long as one side; that is, dividing the expression by 6, the sum of two sides of the square 9, we have the quotient $\frac{61-24x}{24}$ as one of the shorter sides of an equivalent rectangle, the longer of which are 6. Now if one of the longer, (6), be divided by one of the shorter ($\frac{61-24x}{24}$) the quotient will be the number of times the latter is contained in the former, and also the number of times the square of $\frac{61-24x}{24}$ is contained in the area of the equivalent rectangle; that is, we get the expression $\frac{144}{61-24x}$, which represents a number—a ratio. Therefore the square of $\frac{61-24x}{24}$ multiplied by this expression will equal the area aforesaid and we have this equation

$$\left(\frac{61-24x}{24}\right)^2\frac{144}{61-24x}=\frac{61}{4}-6x=\frac{61-24x}{4} \quad . \quad . \quad . \quad (4)$$

a glance at which shows it to be a true one.

Being a true equation, it is evident that were the ratio $\frac{144}{61-24x}$ known, the value of the unknown quantity x could be at once determined. We have no means of knowing this value, but according to the requirement of the Pythagorean law, which in this case is that x^2+x shall equal 9, it becomes $\frac{144}{73-12\sqrt{37}}$. Make this substitution, and the equation becomes

$$\left(\frac{61-24x}{24}\right)^2\frac{144}{73-12\sqrt{37}}=\frac{61-24x}{4}$$

Now this equation gives $\frac{61}{24}$ for the value of x.* Not only this, but *any* value, decimal, fractional, mixed, or whole number, from zero to infinity, substituted for this ratio will give the same value, $\frac{61}{24}$ for x. This could not be possible unless $\frac{61}{24}-x=0$, which means that $x=\frac{61}{24}$, which we see is one of the values that satisfies the equation "(3)" of the third degree, already discussed and solves the riddle of the sum ($\frac{61}{4}-6x+6x-15$) of the quantities that make up the area between the square 9 and the larger rectangle of the hypothenuse of our triangle after having its "square completed." ✕

I have gone into the mathematical details as little as possible to make the subject intelligible. The whole detailed proof of what I have outlined to you I have here with me for examination if desired.*

*See appendix, a careful study of which is invited.

✕ square on the hypothenuse

Than those I have touched upon, there are other proofs—direct proofs—to substantiate the indirect ones embraced in this paper. The value $\frac{61}{24}=x$ makes the length of the little line $\overline{ki}$ in the figure (2) $\frac{1}{24}$; because $x-\frac{5}{2}=\frac{61}{24}-\frac{5}{2}=\frac{1}{24}$ and consequently the area of the little square $\frac{1}{576}$—altho' this area, as algebraically expressed by $\frac{61}{4}-6x$, becomes zero, when the value $\frac{61}{24}$ for x is substituted in it—and the area of the little rectangles $\frac{1}{4}$, for $\frac{1}{24}\times3\times2=\frac{1}{4}$.

A development of the figure (2), on a large scale, seems to fully confirm the value $\frac{1}{24}$, for the line $\overline{ki}$. The conclusion therefore forces itself on us that $x^2+x=9$ is not a true equation—that $\frac{1}{576}$ must be added to the second member to make it so: that is, that $x^2+x=9\frac{1}{576}$. Now if the first member be squared by adding $\frac{1}{4}$, the second member, now its equal, is also squared by adding $\frac{1}{4}$ to it. Thus $x^2+x+\frac{1}{4}=9+\frac{1}{576}+\frac{1}{4}=\frac{5329}{576}$. Extracting the square root of both members of this equation we have

$$x+\tfrac{1}{2}=\tfrac{73}{24} \text{ and } x=\tfrac{73}{24}-\tfrac{1}{2}=\tfrac{61}{24}.$$

From all of which it would appear that for this particular triangle the Pythagorean law is not true—that it is an exception. Not only this, but as there can be constructed an infinite number of such triangles, that it is only true for what have been rightly called the "right angle triangle numbers;" not true for any case in which is involved the extraction of the square root of an imperfect square to obtain the value of what must be a positive line, not a myth whose existence depends on the number of decimals that be allowed to constitute its tail.

I trust this has been made clear to you; that the mathematics sustains the à priori reasoning: that one law of mathematics cannot invalidate another equally vital for the support of truth and that mathematics even may become crystalized into petrifactions of imperfectly understood laws and thus passed from generation to generation to the detriment of truth and the discredit of science.

Now, having reached this conclusion from perfectly

sound premises—having so weakened this cornerstone of mathematics that we see the whole structure of the science tottering to its fall, I beg to vacate the position of essayist for the evening to take my place as a member of the Club to discuss the paper and *to* discuss it from the inside, as it were, before I close.

In doing so I want to point out the value of what in these days is called "The higher criticism."* Things, beliefs, and what not, have been overhauled by it, and discredited, if not altogether relegated to the oblivion of pure fiction. Men are wiser now than formerly—in some things. The higher criticism destroys history, converts facts into myths, unsettles and perverts religious beliefs. Its syllogisms are founded, not on laws or postulates, fixed by mathematical processes of reasoning, but on problematical, half revealed, or totally misunderstood records of past events and on self evolved premises with a basis of common sense. In short, it lacks unassailable conclusions—it's defective in that it cannot begin with postulates that no man can fail to accept, like the postulates of mathematics.

It is said that figures will not lie—cannot be made to lie. Now, I will reveal to you, that the first part of this essay is a mathematical romance, pure and simple! But, it was begun in perfect honesty and with no other object in view than the revelation of truth.

I believed,—why I have clearly told you,—that all these centuries a law had been accepted as universal to the detriment of true science and I set about to demonstrate it. Now, if *I* can enter the domain of mathematics and use its own formulas so successfully to prove that it is wrong, how easy for those who march under the banner of "higher criticism" to assail accepted things not based on such well defined and easily determined foundations as the laws of mathematics, which, like those of the Medes and Persians, "altereth not"?

* The higher criticism meant is *not* that which is the true handmaid of science, a fair sample of which may be found in a recent publication "Genesis and Modern Science."—Warren R. Perce, James Potts & Co., New York.

It seems to me that what I have done with the higher criticism in mathematics would deceive the "very elect." Now, I have not only—and honestly at that—made figures lie, but have brought into open Court two credible witnesses—viz: Algebra and Geometry—who have sworn to the lies of Arithmetic, thus endangering the integrity of a venerable structure, hoary with age and the Pantheon of the mathematical Saints!

In the science of military engineering there are parts or adjuncts of fortifications and field works called *chevaux de frise* and *trench cavaliers*. A teacher once, it is said, in order, it is presumed, to test a student's knowledge, asked him the question: "Suppose the *trench cavalier* should run away, what would you do?" Without hesitation he answered "Mount a *Cheval de frise* and go after him." Now, when the Delia Bacons, Ignatius Donnellys, and others, like myself, for instance, start out with dark lanterns to look over ground, that has been thoroughly gone over by others in broad day light, to search for hidden knowledge, we find stable things in a state of perturbation. The uncertain light presents things dimly; outlines are not sharply defined; anything is possible from the stampeding of the *chevaux de frise* to the flight of the *trench cavalier*. The latter, in this case, was $x^2+x=9$. The darkness distorted it. It was in a state of perturbation. Its position was uncertain—it was, in short, in a state of flight. I could neither catch it nor get rid of it. It lurked in every triangle and hid behind every square encountered in my pursuit of it. The persistency with which it would rise, apparently from the dead, was somewhat appalling. When I believed I was finally burying its corpse, no sooner was the earth filled into the grave, than, lo! there were the remains out of the ground as before and I cried out, not for darkness and Blücher, but for daylight, Georg Cantor and Cauchy!—whose acquaintance you will make later on.

In a publication called "The Monist," issue for October, 1896,—Vol. 7, No. 1, page 21— article, "The regenerated logic," may be found the following, viz:

"It is a remarkable historical fact that there is a "branch of science in which there never has been a pro- "longed dispute concerning the proper objects of that "science. It is the mathematics. Mistakes in mathe- "matics occur not infrequently, and not being detected, "give rise to false doctrine, which may continue a long "time. Thus a mistake in the evaluation of a definite "integral by Laplace, in his '*Mecanique Cèleste*,' led to "an erroneous doctrine about the motion of the moon, "which remained undetected for nearly a century. But "after the question had once been raised, all dispute "was brought to a close within a year. So, several "demonstrations in the first book of Euclid, notably "that in the sixteenth proposition, are vitiated by the "erroneous assumption that a part is necessarily less "than its whole. These remained undetected until "after the theory of the non-Euclidean Geometry had "been completely worked out; but since that time no "mathematician has defended them, nor could any "competent mathematician do so, in view of Georg "Cantor's or Cauchy's discoveries."
Behold the Cantor and the Cauchy! Great must be Georg, or even Cauchy, if he has proved that a part is not necessarily less than the whole of a thing! The audacity of my attacking the Pythagorian law was nothing to this achievement of Georg.

Again, the same publication, same number, subject, "Subconscious Pangeometry," page 100, notices a book from the press of Leubner, Leipsic, "The theory of parallels." "A work which perhaps can best be de- "scribed as a book on 'The Non-Euclidian Geometry' "Inevitable"—"It confers the estimable blessing on "thinkers by giving them the actual documents which "are the slow, groping awakening of the world-mind at "the gradual dawning of what has now become the full "day of self-conscious non-Euclidian Geometry."

Who knows but what I am now marching with the "subconscious" fellows under the banner of Pan-geometry? Who, according to this notice—it gives the bibliography of the subject—are proving that the three

*Teubner

angles of a triangle are not equal to two right angles; that parallel lines are not parallel! When I am conscious that I am enrolled with these illustrious fellows I will again mount a *cheval de frise* and chase the fleeing $x^2+x=9$.

But, in all seriousness, what I believe I have accomplished is, that the little line, *ki*, of our figure (2) is the *unit* in this particular case, and that I can give in any particular case the unit and the ultimate atom of division—the mathematical microbe of exactness. For instance, without going into details, this inconceivable quantity for $x^2+x=9$ is $\frac{1}{12278016}$, the square root of which is $\frac{1}{3504}$, and a value $(\frac{8905}{3504})$ determined for x, is true within $\frac{1}{3504}$ of the unit. When this value is substituted for x in the equation $x^2+x=9$, it gives $x^2+x=9\frac{1}{12278016}$. Squaring the x $(\frac{8905}{3504})$ makes the x^2 but $\frac{1}{12278016}$ too great for the sum to be 9.

Further, I can fix the quantities, areas, whose sum equals 9, viz:

$$\frac{22631}{3504}+\frac{8905}{3504}=9.$$

The first is not the square of the second, and neither is a perfect square, but they are commensurable; their unit is the same, viz: $\frac{1}{3504}$.

APPENDIX.

It is presumed the geometrical figures, explanations, etc., of the foregoing text are sufficiently comprehended to require no repetition. The references below—"fig 1" and "fig 2"—are to the geometrical figures of the text.

It may be objected that some equations used in what follows—for instance (4) of the essay—are indeterminate, and that the first members *reduced* become identical with the second. This is true of equation (4), but identity of members insures their perfect equality to start with, and when a definite value is given to the ratio $\frac{144}{61-24x}$ this identity disappears and the equation ceases to be indeterminate, and can be true but for one value of the variable. In this form only is it discussed; it could not be in any other—so with the others.

The equation (3) of the essay, viz.:

$$24x^3=37x^2+277x-549 \quad . \quad . \quad . \quad (3)$$

as shown by it's construction, has imposed upon it two conditions, that $x^2+x=9$ and $\frac{61}{4}-6x$ shall equal zero. Hence two values, $x=\frac{\sqrt{37}-1}{2}$ and $x=\frac{61}{24}$ satisfy it. The first reduces it to $120\sqrt{37}-336=120\sqrt{37}-336$, the second to $\frac{226981}{576}=\frac{226981}{576}$.

The following equation, viz:,

$$\left.\begin{array}{l} 24x^3=36x^2+276x-540 \\ 2x^3=3x^2+23x-45 \end{array}\right\} \quad . \quad . \quad . \quad (4)$$

or, reducing,

may be obtained from each of these three;

$$\frac{12}{2x-5}(x^2-5x+\tfrac{25}{4})+x^2-5x+\tfrac{25}{4}=\tfrac{1}{4}$$

$$\frac{12}{2x-5}(x^2-5x+\tfrac{25}{4})+\tfrac{61}{4}-64=x^2+x-\tfrac{35}{4}$$

$$\left(\frac{4x^2+4x-35}{4x^2-20x+25}\right)\left(\tfrac{61}{4}-6x\right)=\tfrac{1}{4}$$

which are modifications, authorized by the law, of

$$\frac{12}{2x-5}(x^2-5x+\tfrac{25}{4})+x^2-5x+\tfrac{25}{4}=x^2+x-\tfrac{35}{4} \quad . \quad . \quad . \quad (a)$$

and, $\left(\frac{4x^2+4x-35}{4x^2-20x+25}\right)(x^2-5x+\frac{25}{4})=x^2+x-\frac{35}{4}$ . . . (b)

In these equations, $\frac{12}{2x-5}$ and $\frac{4x^2+4x-35}{4x^2-20x+25}$ are the number of times, respectively, that the little square *l. n. j. o.* is contained in the sum of the two rectangles *k. i. n. l* and *e. m. l. o.* (fig 2), and in this sum *plus* the said square—the sum of the three—viz.: $x^2+x-\frac{35}{4}$, which the law requires to equal $\frac{1}{4}$. That the members of equations (a) and (b) are *equal* is apparent; and, also, how the three preceding ones are obtained from them.

Equation (4) is, like equation (3), remarkable, in that two values, $x=\frac{\sqrt{37}-1}{2}$ and $x=\frac{5}{2}$ satisfy it. The first reduces it to $10\sqrt{37}-28=10\sqrt{37}-28$, the second to $\frac{125}{4}=\frac{125}{4}$. It is possible for x to equal either $\frac{\sqrt{37}-1}{2}$ or $\frac{61}{24}$, but impossible for it to equal $\frac{5}{2}$; for this value would make the line $\overline{ki}$ (fig. 2) zero, and consequently $x^2-5x+\frac{25}{4}$, zero. It would be the *reductio ad absurdum* unless it be interpreted to mean that the value $x=\frac{5}{2}$ satisfies equation (4) because $x^2-5x+\frac{25}{4}=0$ is, under the law, (the application of which produces this impossible value), but $\frac{61}{4}-6x=0$. The law requires $x-\frac{5}{2}$ to equal $\sqrt{\frac{61}{4}-6x}$. If the former be zero for any value of x the latter must be also; therefore their squares are equal.

We have then two equations, the first members of which are identical, but the second differ by x^2+x-9; thus,

$$24x^3=37x^2+277x-549 \quad . \quad . \quad . \quad (3)$$
$$\underline{24x^3=36x^2+276x-540} \quad . \quad . \quad . \quad (4)$$
$$x^2+x-9$$

Any number of times x^2+x-9 added to the second member of the first (3) will not affect its value for $x=\frac{\sqrt{37}-1}{2}$. So also any number of times $x^2+x-9\frac{1}{576}$ added to it will not affect its value for $x=\frac{61}{24}$. Any number of times x^2+x-9 added to the second member of the latter (4) will not affect its value for $x=\frac{\sqrt{37}-1}{2}$; so also any number of times $x-\frac{5}{2}$, or $x^2+x-\frac{35}{4}$, added to it will not affect its value for $x=\frac{5}{2}$. Which is correct, $x^2+x-\frac{35}{4}=0$, $x^2+x-9=0$, or $x^2+x-9\frac{1}{576}=0$; are these merely eccentricities of cubic equations?

Take the equation (4) of the essay, viz:

$$\left(\frac{61-24x}{24}\right)^2 \cdot \frac{144}{61-24x}=\frac{61}{4}-6x$$

A glance shows it true for any value of x, but were the ratio $\frac{144}{61-24x}$ known it could be but for one. Now the law requires this ratio to be $\frac{144}{73-12\sqrt{37}}$; substitute it and the equation becomes

$$\left(\frac{61-24x}{24}\right)^2 \cdot \frac{144}{73-12\sqrt{37}} = \frac{61}{4} - 6x = \frac{61-24x}{4} \quad . \quad . \quad . \quad (5)$$

It would appear that this equation must give the value for x demanded by the law, for the ratio has been given the only value it can have under the law. So it will, if the factor $61-24x$, common to both members, be divided out, and in like manner the equation will give any value for x. But why not, with the common factor left in? No true equation is affected by multiplying both members of it by anything: that is, the equality of its members cannot be affected: therefore it must follow that

$$\left(\frac{61-24x}{576}\right) \cdot \frac{144}{73-12\sqrt{37}} = \frac{1}{4}$$

(which equation (5) becomes after dividing out the common factor) is false; hence $x^2+x=9$, on which it depends for the equality of its members, is false. Solved with the common factor left in, the value $x=\frac{61}{24}$ is found; this makes the common factor zero. This result is consistent, for the members of a false equation multiplied by an expression, the value of which is zero, reduces both to zero and therefore to an equality.

Solving equation (5).

$$\frac{(61-24x)^2}{576} = \frac{(61-24x)(73-12\sqrt{37})}{576}$$

$$(61-24x)^2 = (61-24x)(73-12\sqrt{37})$$

$$3721 - 2928x + 576x^2 = 4453 - 732\sqrt{37} - 1752x + 288x\sqrt{37}$$

$$576x^2 - 1176x - 288x\sqrt{37} = 732 - 732\sqrt{37}$$

$$x^2 - \frac{1176+288\sqrt{37}}{576}x = \frac{732-732\sqrt{37}}{576}$$

completing squares

$$x^2 - \frac{1176+288\sqrt{37}}{576}x + \left(\frac{588+144\sqrt{37}}{576}\right)^2$$
$$= \frac{732-732\sqrt{37}}{576} + \left(\frac{588+144\sqrt{37}}{576}\right)^2$$

reducing the *second* member of this last equation

$$\frac{732-732\sqrt{37}}{576} + \left(\frac{588+144\sqrt{37}}{576}\right)^2$$
$$= \frac{576(732-732\sqrt{37})}{(576)^2} + \left(\frac{588+144\sqrt{37}}{576}\right)^2$$
$$= \frac{421632-421632\sqrt{37}}{(576)^2} + \frac{345744+169344\sqrt{37}+767273}{(576)^2}$$

$$=\frac{767376-252288\sqrt{37}+767273}{(576)^2}$$

$$=\left(\frac{876-144\sqrt{37}}{576}\right)^2$$

Hence, $x^2-\frac{1176+288\sqrt{37}}{576}x+\left(\frac{588+144\sqrt{37}}{576}\right)^2=\left(\frac{876-144\sqrt{37}}{576}\right)^2$

extracting square root of both members

$$x-\frac{588+144\sqrt{37}}{576}=\frac{876-144\sqrt{37}}{576}$$

or, $\quad x-\frac{588}{576}-\frac{144\sqrt{37}}{576}=\frac{876}{576}-\frac{144\sqrt{37}}{576}$

hence $\quad x=\frac{876+588}{576}=\frac{1464}{576}=\frac{61}{24}$

In like manner any value, from zero to infinity, for x substituted in the ratio, or, which is the same thing, any value given to the ratio, will give the value $x=\frac{61}{24}$. One example will suffice: choosing a value at random, make $\frac{144}{61-24x}=99\frac{3}{7}=\frac{696}{7}$, the equation becomes

$$\left(\frac{61-24x}{24}\right)^2\frac{696}{7}=\frac{61}{4}-6x$$

$$\left(\frac{3721-2928x+576x^2}{576}\right)\frac{696}{7}=\frac{61}{4}-6x=\frac{61-24x}{4}$$

$$\frac{3721-2928x+576x^2}{576}=\left(\frac{61-24x}{4}\right)\frac{7}{696}$$

$$3721-2928x+576x^2=(61-24x)\frac{1008}{696}=(61-24x)\frac{126}{87}$$

$$(3721-2928x+576x^2)87=(61-24x)126$$

$$323727-254736x+50112x^2=7686-3024x$$

$$316041-251712x+50112x^2=0$$

$$x^2-\frac{251712}{50112}x=-\frac{316041}{50112}$$

completing squares

$$x^2-\frac{251712}{50112}x+\left(\frac{125856}{50112}\right)^2=-\frac{316041}{50112}+\frac{15839732736}{(50112)^2}$$

$$=\frac{15839732736-15837446592}{(50112)^2}$$

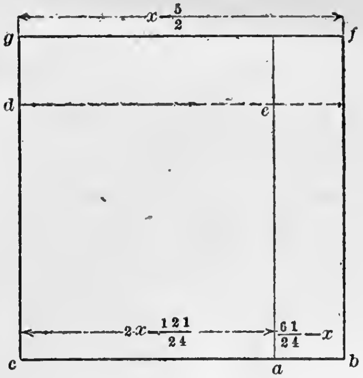
$$=\frac{2286144}{(50112)^2}$$

extracting square root of both members

$$x-\frac{125856}{50112}=\frac{1512}{50112}$$

hence $\quad x=\frac{1512+125856}{50112}=\frac{61}{24}$

Let the following figure represent the little square *l n j o* of fig. 2, enlarged:



Now $\frac{61}{24}-x$ must, under the Pythagorean law, have a value—cannot be zero. Let $\underline{ab}$ of this figure represent—equal—it. $\underline{cb}$ being $x-\frac{5}{2}$ and $\underline{ab}$ $\frac{61}{24}-x$, $\underline{ca}$ will equal $x-\frac{5}{2}-(\frac{61}{24}-x)=2x-\frac{121}{24}$, and the square $c\ a\ e\ d$, $(2x-\frac{121}{24})^2,=4x^2-\frac{121}{6}x+\frac{14641}{576}$.

A part cannot equal the whole; the whole square, $c\ b\ f\ g$, cannot equal $c\ a\ e\ d$, a part of it, so long as $\underline{ab}$ has any value. But let the expressions for the two squares be equated and the equation solved, thus:

$$4x^2-\frac{121}{6}x+\frac{14641}{576}=x^2-5x+\frac{25}{4} \quad . \quad . \quad . \quad (6)$$

$$3x^2-\frac{91}{6}x=-\frac{11041}{576}$$

$$x^2-\frac{91}{18}x=-\frac{11041}{1728}$$

$$x^2-\frac{91}{18}x+(\frac{91}{36})^2=-\frac{11041}{1728}+\frac{8281}{1296}=\frac{33124-33123}{5184}=\frac{1}{5184}$$

hence
$$x-\frac{91}{36}=\frac{1}{72}$$

$$x=\frac{91}{36}+\frac{1}{72}=\frac{183}{72}=\frac{61}{24}.$$

For this value, $x-\frac{5}{2}=\frac{1}{24}$, and $x^2-5x+\frac{25}{4}=\frac{1}{576}=$the first member of equation (6), hence

$$4x^2-\frac{121}{6}x+\frac{14641}{576}=\frac{1}{576}$$

$$x^2-\frac{121}{24}x+\frac{14640}{2304}$$

$$x^2-\frac{121}{24}x+(\frac{121}{48})^2=-\frac{14640}{2304}+\frac{14641}{2304}=\frac{1}{2304}$$

$$x-\frac{121}{48}=\frac{1}{48}$$

$$x=\frac{122}{48}=\frac{61}{24},$$

thus, (equation (6),) $3x^2-\frac{91}{6}x+\frac{11041}{576}=0$; that is, the first member being the difference between $\overline{cb}^2$ and $\overline{ca}^2$, the difference is zero, and consequently $\frac{61}{24}-x=0$.

Now if $\frac{61}{4}-6x=0$,

$$3x^2-\frac{91}{6}x+\frac{11041}{576}=\frac{61}{4}-6x$$

$$x^2-\frac{55}{18}x=-\frac{11041+8784}{1728}=-\frac{2257}{1728}$$

$$x^2-\frac{55}{18}x+(\tfrac{55}{36})^2=-\frac{2257}{1728}+\frac{3025}{1296}=\frac{5329}{5184}$$

$$x-\frac{55}{36}=\frac{73}{72}$$

$$x=\frac{73}{72}+\frac{55}{36}=\frac{61}{24}.$$

Let $\frac{61}{4}-6x$, the equal under the law of the second member of equation (6), replace it, thus:

$$4x^2-\frac{121}{6}x+\frac{14641}{576}=\frac{61}{4}-6x$$

$$x^2-\frac{85}{24}x=-\frac{5857}{2304}$$

$$x^2-\frac{85}{24}x+(\tfrac{85}{48})^2=-\frac{5857}{2304}+\frac{7225}{2304}=\frac{1368}{2304}$$

$x-\frac{85}{48}=\frac{1}{48}\sqrt{1368}$, hence $x=\frac{85+\sqrt{1368}}{48}$, a value a little greater, as it should be, than $x=\frac{\sqrt{37}-1}{2}$, but less than $\frac{61}{24}$.

Therefore $\frac{61}{4}-6x$, apparently, is not equal to $x^2-5x+\frac{25}{4}$, and that the latter is equal to the square of $\underline{ca}$, a part of itself, is because the difference $(3x^2-\frac{91}{6}x+\frac{11041}{576})$ between $\overline{cb}^2$ and $\overline{ca}^2$ equals zero, and consequently $\frac{61}{24}=0$.

If $\frac{61}{24}-x$, $\frac{61}{4}-6x$, and $x^2+x-9\frac{1}{576}$ equal zero, then must

$$\frac{61}{4}-6x+(\tfrac{61}{24}-x)^2=0 \quad . \quad . \quad . \quad (c)$$

and $\frac{61}{4}-6x+(\frac{61}{24}-x)^2=x^2+x-9\frac{1}{576}$ . . . (d).

Solving the first (c):

$$\frac{61}{4}-6x+\frac{3721}{576}-\frac{61}{12}x+x^2=0$$

$$\frac{12505}{576}-\frac{133}{12}x+x^2=0$$

$$x^2-\frac{133}{12}x=-\frac{12505}{576}$$

$$x^2-\frac{133}{12}x+(\tfrac{133}{24})^2=-\frac{12505}{576}+\frac{17689}{576}=\frac{5184}{576}$$

$$x-\frac{133}{24}=\pm\frac{72}{24}$$

$$x=+\frac{205}{24} \text{ and } -\frac{61}{24}.$$

The *plus* value is impossible, but the minus is correct.

Solving the second (d):

$$\tfrac{61}{4}-6x+\tfrac{3721}{576}-\tfrac{61}{12}x+x^2=x^2+x-\tfrac{5185}{576}$$

$$\tfrac{17690}{576}-\tfrac{145}{12}x=0$$

$$\tfrac{145}{12}x=\tfrac{17690}{576} \text{ and } x=\tfrac{17690}{6960}=\tfrac{61}{24}.$$

The above appears to be a cumulative proof that x^2+x does not equal 9, and, so far as algebraic analysis goes, conclusive. Opposed to it is the geometrical proof embodied in the figure 1 of the essay. If this, the geometrical proof, cannot be accepted, it would appear that *all* basis for mathematical reasoning is removed—that there is nothing to start from. If the area of a triangle is not absolutely equal to its base multiplied by half its altitude, there would appear to be no basis for determining the numerical value of any surface—no basis in fact for the algebraic analysis above.

The proof of figure 1 depends, first, on showing that the square on the hypothenuse of a right angle triangle equals exactly four times the area of the triangle plus the square of the difference between its base and altitude; and, secondly, on showing that the sum of the areas of the four equal triangles and the square of the said difference equals the sum of the squares of the base and altitude. The first is evident at a glance; the second requires but a moment's mental figuring. Yet the algebraic analysis confirms in more ways than one the à priori reasoning—the "common sense"—of the essay. Incommensurability was not admitted, and, consequently an impossible line for the hypothenuse or other side of a right angle triangle. If a triangle be a triangle at all, its sides must be positive and definite, and therefore determinable—*commensurable* with something in existence. This is only another way of stating that one member of an equation cannot be either infinitely less or greater than the other—that the equality of its members must be perfect. It is significant that,

in the above analysis, whenever the equation is a true one, that which "squares" one member, always squares the other.

If it can be proved that lines which are *postulated* to be parallel, are not; that the sum of the three angles of a triangle is not equal to 180°; and that "a part is not necessarily less than its whole," then it would appear not so strange, even in the face of the simple, satisfying, and positive proof of the figure 1, that the Pythagorean law may be defective—not general—and only true for the "right angle triangle numbers."

ADDENDUM.

The sum $\frac{61}{4}-6x+6x-15=\frac{1}{4}$ (see page 17)—true for any value of x, and the exclusive creation of the Pythagorean law—furnishes an equation of the 2nd degree (by substituting in it for $\frac{61}{4}-6x$ its equivalent $(\frac{61-24x}{24})^2\frac{144}{61-24x}$, and for the denominator, $61-24x$, of the ratio, the value $73-12\sqrt{37}$, demanded by this law) to which the objections that may be urged to cube roots and the equation (5), page 27, (which, with the factor, $61-24x$, common to both members left in, gives for any value of this ratio $\frac{61}{24}$ for x, but $\frac{\sqrt{37}-1}{2}$ when the ratio is made what the law demands and the common factor divided out) cannot apply, viz:

$$\left(\frac{61-24x}{24}\right)^2\frac{144}{73-12\sqrt{37}}+6x-15=\frac{1}{4}$$

The *sum*, from which this equation is obtained, is unquestionably equal to $\frac{1}{4}$, and $(\frac{61-24x}{24})^2\frac{144}{61-24x}$ is unquestionably the equal of $\frac{61}{4}-6x$, for which it is substituted in this sum. Thus no change has been made and *the sum is still what the Pythagorean law has made it*. Therefore the further substitution, in the ratio, for $61-24x$ the value $73-12\sqrt{37}$ which the law requires, still further impresses on this sum the conditions of the Pythagorean law, and the resulting equation—of the 2nd degree, having no common factor in its members and incapable of giving more than one value for its unknown quantity—should give for x the value demanded by this law: but it does not. This equation reduces to

$$x^2-\frac{1176+288\sqrt{37}}{576}x=\frac{732-732\sqrt{37}}{576}$$

which, it is seen, is identical with the equation on page